Senior Mathematical Challenge

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Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	C	B	B	D	A	D	B	B	E	A	E	D	A	E	C	E	C	D	C	D	B	A

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

Solution **C**

The prime factorisation of 2023 is $7 \times 17 \times 17$ so $\sqrt{\dfrac{2023}{2+0+2+3}} = \sqrt{\dfrac{2023}{7}} = \sqrt{17^2} = 17$.

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

Solution **C**

The difference between one third and 0.333 is $\dfrac{1}{3} - \dfrac{333}{1000} = \dfrac{1000-999}{3000} = \dfrac{1}{3000}$.

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

Solution **D**

The new area = the old area $\times 1.2 \times 0.85$ = the old area $\times 1.02$. This represents a 2% increase.

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

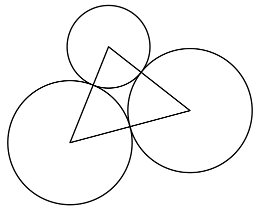
Solution **C**

The world record of 5000 m in 19 minutes and 6 seconds $\simeq$ 5000 m in 20 minutes = 15000 m in 60 minutes = 15000 m in an hour = 15 km/h.

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

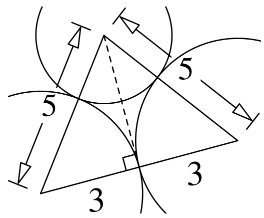
What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18



Solution **B**

The triangle formed by joining the centres of the circles is isosceles, so splitting it along its line of symmetry gives us two right-angled triangles each with a base of 3 and a hypotenuse of 5. Using Pythagoras' Theorem the perpendicular height is 4. The area of the whole triangle is then $\frac{1}{2} \times 6 \times 4 = 12$.



6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Solution **B**

The sum of any three integers in arithmetic progression is a multiple of 3. For proof of this, if we let the smallest integer be a and the common difference of the sequence be d, then $a + (a + d) + (a + 2d) = 3a + 3d = 3(a + d)$. As a result of the way the grid is filled, all the horizontal, vertical and diagonal lines contain numbers which are in arithmetic progression. Horizontally there are 2 lines of three cells in each of the 4 rows. Here $d = 1$. Vertically, there are again 2 lines in each of the 4 columns. Here $d = 4$. On the diagonals with positive gradient, there are 4 lines, with $d = -3$. On the diagonals with negative gradient there are four lines with $d = 5$. This is a total of $8 + 8 + 4 + 4 = 24$ lines.

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25th term?

A 2010 B 2009 C 2008 D 2007 E 2006

Solution **D**

The sequence begins 2023, 2022, 1, 2021, 2020, 1, 2019, 2018, 1 ... . Let the k^{th} term be u_k. Now consider the sequence $u_1, u_4, u_7, \ldots$, which starts 2023, 2021, 2019, ... Here the terms decrease by two each time. Since $25 = 1 + 8 \times 3$, $u_{25} = u_1 - 8 \times 2 = 2023 - 16 = 2007$.

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

SOLUTION **A**

The value of $99(0.\dot{4}\dot{9} - 0.\dot{4}) = 99\left(\frac{49}{99} - \frac{4}{9}\right) = 99\left(\frac{49}{99} - \frac{44}{99}\right) = 99\left(\frac{49 - 44}{99}\right) = 99 \times \frac{5}{99} = 5$.

9. When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

Across
1. A square
3. A fourth power

Down
1. Twice a fifth power
2. A cube

1	*	2
3		

A 0 B 1 C 2 D 4 E 6

SOLUTION **D**

For 1 Down, $2 \times 2^5 = 64$ is too small and $2 \times 4^5 = 2048$ is too big and therefore we must have $2 \times 3^5 = 486$. 3 Across must then start with a 6 and is therefore $5^4 = 625$. 2 Down must then end in a 5 and is therefore $5^3 = 125$. 1 Across is then 4 * 1. The only square of this form is $21^2 = 441$, so * is a 4.

10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

The sum $2^{2024} + 2^{2023} + 2^{2022}$ can be factorised to $2^{2022}(2^2 + 2^1 + 1) = 2^{2022} \times 7$. Hence, of the numbers listed, only 7 and $8 = 2^3$ are factors of 2^{2022}.

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

SOLUTION **B**

Each of the four people is either telling the truth or lying. Assume first that Wenlu is telling the truth, then Xander is lying, which implies that Yasser is telling the truth which finally implies that Zoe is also telling the truth. In this case 3 people tell the truth. Now assume that Wenlu is lying. Therefore Xander is telling the truth that Yasser is lying and finally Zoe is also lying. In this case only 1 person tells the truth. In both cases, all four statements are consistent with each other.

12. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n.)$

What is k?

A 4 B 5 C 6 D 7 E 8

SOLUTION **E**

As $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$, factors of 50! which contain a factor of 7 are 7, 14, 21, 28, 35, 42 and 49. The first six of these each contribute a single factor of 7 and 49 contributes two. The greatest power of 7 which is a factor of 50! is then 7^8, so $k = 8$.

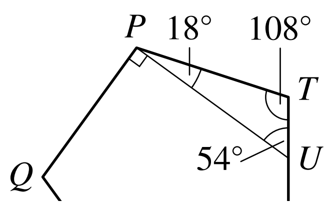
13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A $1:3:6$ B $1:2:4$ C $2:3:4$ D $1:4:8$ E $1:3:5$

SOLUTION **A**

The interior angles of a regular pentagon are $180° - \frac{360°}{5} = 108°$. As $\angle QPU$ is a right angle, $\angle UPT = 108° - 90° = 18°$. As angles in a triangle sum to $180°$, $\angle PUT = 180° - (108° + 18°) = 54°$. Therefore $\angle TPU : \angle PUT : \angle UTP = 18 : 54 : 108 = 1 : 3 : 6$.



14. The points $P\ (d, -d)$ and $Q\ (12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

A -16 B -4 C 4 D 8 E 16

Solution **E**

The equation of the circle is $x^2 + y^2 = r^2$. At Q, $(12 - d)^2 + (2d - 6)^2 = r^2$. At P, $d^2 + (-d)^2 = r^2$, so $2d^2 = r^2$. Expanding the first equation and subtracting the second gives $144 - 24d + d^2 + 4d^2 - 24d + 36 - 2d^2 = 0$, which simplifies to $3d^2 - 48d + 180 = 0$. Dividing by 3 and factorising gives $(d - 6)(d - 10) = 0$. Therefore $d = 6$ or $d = 10$ and the sum of these values is 16.

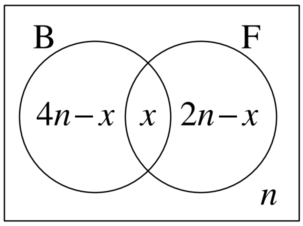
15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

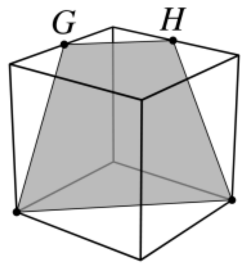
A 19 B 14 C 9 D 5 E 0

Solution **D**

Let the number of people who play both basketball and football be x and the number who play neither be n. A Venn diagram can then be filled as shown. As there are 30 students, $7n - x = 30$. As $x \geq 0$, $7n - 30 \geq 0$ and so $n \geq 5$. From the Venn diagram it can be seen that $2n - x \geq 0$, therefore $2n - (7n - 30) \geq 0$ so $n \leq 6$. So $n = 5$ or 6 and the corresponding values of x are 5 or 12. The only one of these in the listed options is $x = 5$.



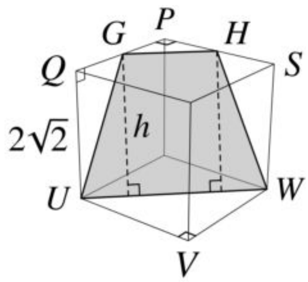
16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.



What is the area of the trapezium?

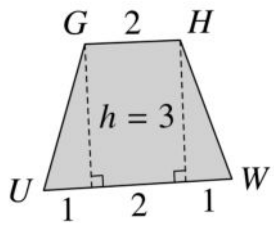
A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$

Solution **A**



To find the area of the trapezium, we require lengths of GH, UW and the perpendicular distance between them, h, say. In triangle PGH, $PG = PH = \sqrt{2}$ therefore $GH = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$.

Triangle VUW is an enlargement of triangle PGH with scale factor 2, so $UW = 4$. In order to find h we must first find length GU. In triangle QUG, $UG = \sqrt{(2\sqrt{2})^2 + \sqrt{2}^2} = \sqrt{10}$.



From the triangular end of the trapezium, it follows that $1^2 + h^2 = \sqrt{10}^2$ therefore $h = 3$. The area of the trapezium $= \frac{1}{2} \times (4+2) \times 3 = 9$.

17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

Solution **E**

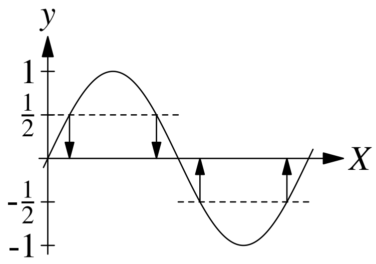
In order to get the largest number, N, we need to make its earlier digits as large as possible, starting 9876 ... as far as this works. However, since 53, 43, 23 and 13 are all prime, the digit 3 must precede all of 5, 4, 2 and 1. So the latest 3 can come is immediately after 6. Thereafter there are no reasons not to follow numerical order, making $N = 987635421$. Its 5^{th} and 6^{th} digits are 3 and 5.

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

A 1 B 2 C 4 D 6 E 8

SOLUTION **C**

The equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ factorises to give $(1 + 2\sin X) - 4\sin^2 X(1 + 2\sin X) = 0$ and then to $(1 + 2\sin X)(1 - 4\sin^2 X) = 0$. Fully factorised, we have $(1 + 2\sin X)(1 + 2\sin X)(1 - 2\sin X) = 0$. So $\sin X = -\frac{1}{2}$ or $\sin X = \frac{1}{2}$. For $0° < X < 360°$, there are then four solutions as shown in the diagram.



19. The expression $\dfrac{7n + 12}{2n + 3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

SOLUTION **E**

The expression $\dfrac{7n + 12}{2n + 3} \equiv \dfrac{4(2n + 3)}{2n + 3} - \dfrac{n}{2n + 3} \equiv 4 - \dfrac{n}{2n + 3}$. The first expression takes integer values precisely when $\dfrac{n}{2n + 3}$ is an integer.

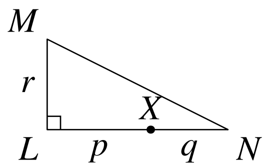
Consider first $n > 0$. When $n > 0$, $2n + 3 > n$, therefore $\dfrac{n}{2n + 3} < 1$ so no integer values of the expression are possible.

Next, consider $n = 0$. In this case, $\dfrac{n}{2n + 3} = \dfrac{0}{0 + 3} = 0$ which is an integer.

When $n < 0$, in order to form an integer, we require $n \le 2n + 3$, therefore $n \ge -3$.

Possible values of n are then $n = -1$, -2 and -3. The values of $\dfrac{n}{2n + 3}$ in these cases are $\dfrac{-1}{2 \times (-1) + 3} = -1$, $\dfrac{-2}{2 \times (-2) + 3} = 2$ and $\dfrac{-3}{2 \times (-3) + 3} = 1$. Therefore the sum of the integer values of the initial expression is $(4 - 0) + (4 - (-1)) + (4 - 2) + (4 - 1) = 14$.

20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



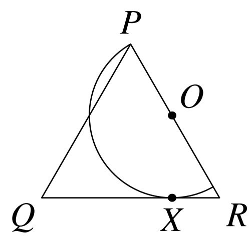
Which of these expressions gives q in terms of p and r?

A $\frac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \frac{p}{2}$ C $\frac{pr}{2p + r}$ D $\frac{p}{2}$
E 1

Solution **C**

Using Pythagoras' Theorem, $NM = \sqrt{(p+q)^2 + r^2}$ so the two journeys have lengths $p + r$ and $q + \sqrt{(p+q)^2 + r^2}$. Equating and rearranging, $p + r - q = \sqrt{(p+q)^2 + r^2}$ and so $(p+r-q)^2 = (p+q)^2 + r^2$. Expanding leads to $p^2 + 2pr - 2pq + r^2 - 2qr + q^2 = p^2 + 2pq + q^2 + r^2$ and therefore $2pr - 2qr = 4pq$. Rearranging to give q in terms of p and r, $pr = q(2p + r)$ so $q = \frac{pr}{2p + r}$.

21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.

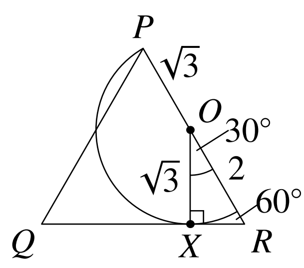


What is the length of QX?

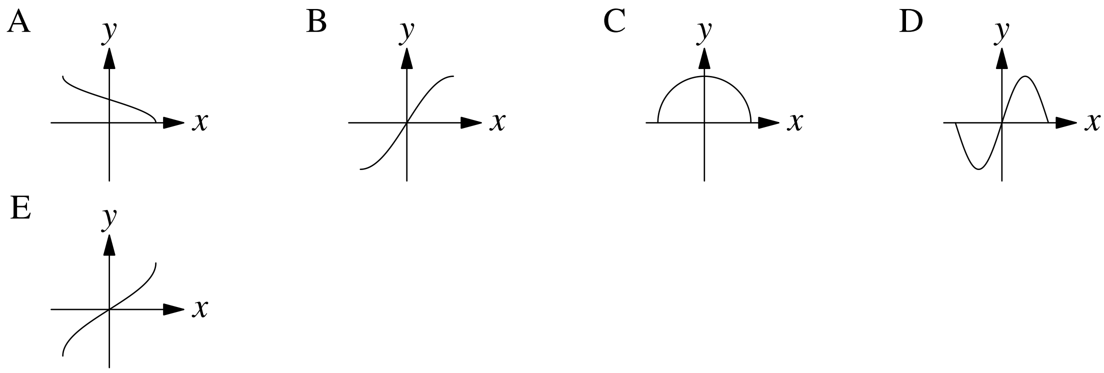
A $\sqrt{3}$ B $2 - \sqrt{3}$ C $2\sqrt{3} - 1$ D $1 + \sqrt{3}$
E $2\sqrt{3}$

Solution **D**

As the semicircle touches QR at X, the radius OX and tangent QR are perpendicular as shown. Triangle OXR is a 30°, 90°, 60° triangle and OX is given as $\sqrt{3}$. Therefore $XR = 1$ and $OR = 2$. As OP is also a radius of the circle, $OP = \sqrt{3}$ and $PR = QR = 2 + \sqrt{3}$. The length $QX = (2 + \sqrt{3}) - 1 = 1 + \sqrt{3}$.



22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?



Solution **C**

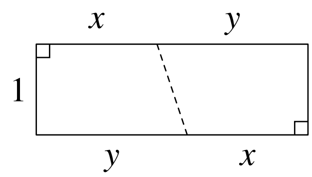
Let $z = (\cos^{-1} x)$. Then $x = \cos z$ and $y = \sin z$ and therefore $x^2 + y^2 = 1$. As z lies between $0°$ and $180°$, x lies between -1 and 1 and y lies between 0 and 1. Hence we get the upper semicircle shown on the graph in option C.

23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

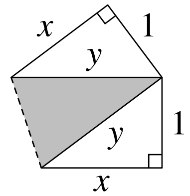
What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\frac{2}{3}$ B $\frac{11}{16}$ C $\frac{12}{17}$ D $\frac{13}{18}$ E $\frac{14}{19}$

SOLUTION **D**



In the first diagram shown, the paper is to be folded so that the bottom left vertex will lie on top of the top right vertex in order to form the desired pentagon. The fold line, shown dotted, must therefore lie on the perpendicular bisector of the line joining the bottom left and top right vertices and so pass through the centre of the rectangle.



Labelling the longest sides of the rectangle with x and y,

$$x + y = 3. \quad (1)$$

From the folded diagram, we have two right-angled triangles and in each, $1 + x^2 = y^2$. Rearranging and factorising gives

$$1 = (y + x)(y - x). \quad (2)$$

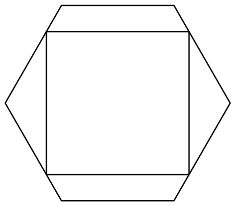
Substituting (1) into (2) gives $1 = 3(y - x)$ and so

$$\tfrac{1}{3} = y - x. \quad (3)$$

Solving (1) and (3) leads to $y = \frac{5}{3}$ and $x = \frac{4}{3}$. The area of the pentagon $= 1 \times 3 -$ the shaded area. As the shaded area can be viewed as a triangle with base y and therefore perpendicular height 1, the area of the pentagon $= 3 - \frac{1}{2} \times y \times 1 = 3 - \frac{1}{2} \times \frac{5}{3} \times 1 = \frac{13}{6}$.

The area of the pentagon as a fraction of the area of the original rectangle is $\dfrac{\frac{13}{6}}{3} = \dfrac{13}{18}$.

24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.



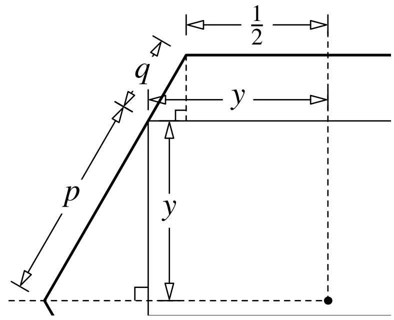
What is the length of the sides of the square?

A $\frac{5}{4}$ B $3 - \sqrt{3}$ C $\frac{4}{3}$ D $\sqrt{2}$ E $\frac{3}{2}$

SOLUTION **B**

Let the square have side-length $2y$.
The two triangles shown, with hypotenuses p and q, have angles 30°, 60° and 90°. As the hexagon has side-length 1,

$$p + q = 1. \qquad (1)$$



From the larger triangle and from the top left of the square,

$$y = \frac{\sqrt{3}p}{2} \qquad \text{and} \qquad y = \tfrac{1}{2}q + \tfrac{1}{2}. \qquad (2)$$

Equating the two equations in (2) and rearranging gives $\sqrt{3}p - q = 1.$ (3)

Solving (1) and (3) simultaneously gives $(\sqrt{3} + 1)p = 2.$

Rearranging and rationalising leads to

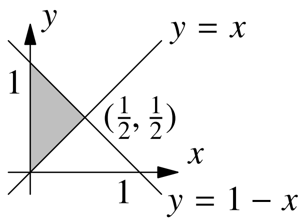
$$p = \frac{2}{(\sqrt{3} + 1)} \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)} = \sqrt{3} - 1.$$

Therefore, the length of the side of the square $2y = \sqrt{3}(\sqrt{3} - 1) = 3 - \sqrt{3}$.

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\frac{1}{4}$ B $\frac{1}{2}$ C 1 D 2 E 4

Solution **A**



Factorising $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ gives $xy^2(x^2 - x - y^2 + y) \geq 0$. Rearranging to $xy^2(y - x - (y^2 - x^2)) \geq 0$ and then factorising gives $xy^2(y - x)(1 - y - x) \geq 0$. As $0 \leq x \leq y$, we know that $x \geq 0$, $y^2 \geq 0$ and $(y - x) \geq 0$ so the fourth factor, $(1 - y - x) \geq 0$. This rearranges to $y \leq 1 - x$. The lines $y = x$ and $y = 1 - x$ meet at $(\frac{1}{2}, \frac{1}{2})$ so the shaded region has area $\frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$.